MA3210 Mathematical Analysis I II

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Contents

References

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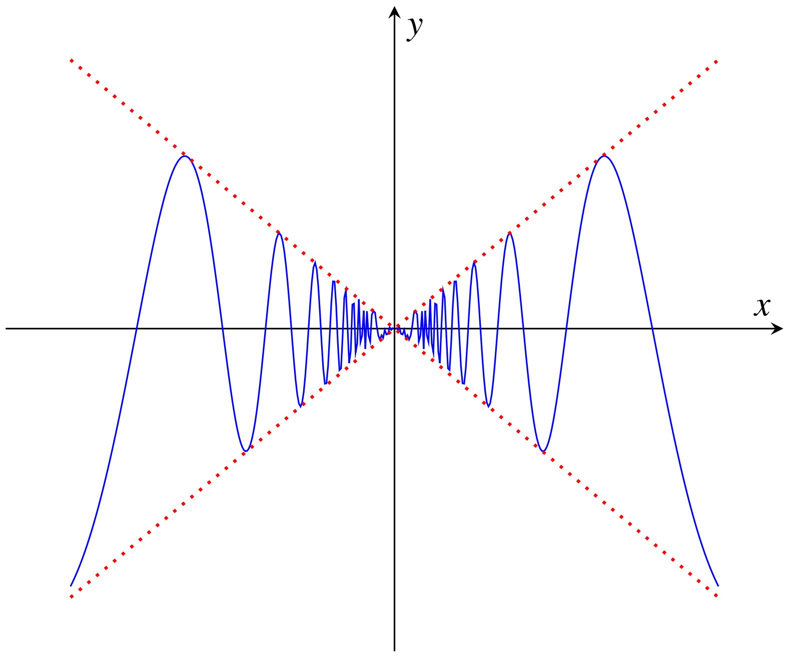

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Chapter 1
Differentiable Functions

1.1
First Principles

Definition 1.1 (differentiable function)**.** A function f is differentiable at a point a if f is defined in some open interval containing a and the limit

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\quad\text{exists.}$$

In this case, $f'(a)$ is the derivative of f at $x=a$.

Geometrically, $f'(a)$ is the slope of the tangent to the curve $y=f(x)$ at $x=a$. The formula in Definition 1.1 is similar to one that students have learnt in H2 Mathematics, which is the derivative of $f(x)$, denoted by $f'(x)$, can be expressed as

$$\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}.$$

Example 1.1 (MA2108 AY22/23 Sem 1 Warmup)**.** Let $y=f(x)$ be continuous everywhere for $x\in(-\infty,\infty)$ and satisfy

$$f(0)=1, f(1)=e \text{ and } f(x+y)=f(x)f(y).$$

Prove that $f(x)=e^x$ for $x\in(-\infty,\infty)$.

Solution. It is clear that

$$f\left(\sum_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} f(x_i).$$

By first principles,

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{f(x)f(\delta x) - f(x)}{\delta x} = f(x) \lim_{\delta x \to 0} \frac{f(\delta x) - 1}{\delta x} = f(x)f'(0).$$

We set $c = f'(0)$ so $f'(x) = cf(x)$. Integrating, we have $\ln|f(x)| = cx + d$. Since $f(0) = 1$, then $d = 0$. Since $f(1) = e$, then $c = 1$. Thus, $\ln|f(x)| = x$. As such, we conclude that $f(x) = e^x$. □

Definition 1.2. If f is differentiable at every point in (a,b), then f is differentiable on (a,b).

Proposition 1.1. If the function $f : [a,b] \to \mathbb{R}$ is such that f is differentiable on (a,b) and the one sided limits

$$L_1 = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad L_2 = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

exist, then f is differentiable on $[a,b]$. In this case, $f'(a) = L_1$ and $f'(b) = L_2$.

1.2 Continuity and Differentiability

Definition 1.3. f is continuously differentiable on I if f is differentiable on I and f' is continuous on I.

Definition 1.4. The collection of all functions which are continuously differentiable on I is denoted by $\mathcal{C}^1(I)$.

Proposition 1.2. If f is differentiable at a, then it is continuous at a.

Proof. We have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} (f(x) - f(a)) + \lim_{x \to a} f(a) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right) + f(a) \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \to a} x - a \right) + f(a) \\
&= f'(a) \cdot 0 + f(a)
\end{aligned}$$

which is just $f(a)$. □

The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere. It is an example of a fractal curve named after its discoverer German mathematician Karl Weierstrass[†].

[†] This link provides an analysis of the Weierstrass function involving its uniform convergence (this term will be studied in due course) and it being nowhere differentiable. This involves the Weierstrass M-test.

Definition 1.5 (Weierstrass function)**.** In Weierstrass's original paper, the function was defined as the following Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is a positive odd integer and $ab > 1 + 3\pi/2$.

Theorem 1.1 (Carathéodory's theorem)**.** Let I be an interval, $f : I \to \mathbb{R}$ and $c \in I$. Then $f'(c)$ exists if and only if there exists a function ϕ on I such that ϕ is continuous at c and

$$f(x) - f(c) = \phi(x)(x-c) \quad \text{for all } x \in I.$$

Proposition 1.3 (chain rule)**.** Let I and J be intervals, and let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be such that $f(J) \subseteq I$. If $a \in J$, f is differentiable at a and g is differentiable at $f(a)$, then $h = g \circ f$ is differentiable at a, and

$$h'(a) = g'(f(a))f'(a).$$

Theorem 1.2 (inverse function theorem)**.** If f is a continuously differentiable function with non-zero derivative at a; then f is invertible in a neighbourhood of a, the inverse is continuously differentiable, and the derivative of the inverse function at $b = f(a)$ is the reciprocal of the derivative of f at a. As an equation, we have

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

1.3 Mean Value Theorem and Applications

Definition 1.6. Let I be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$.

- **(i)** If $f(x_0) \geq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute maximum of f on I
- **(ii)** If $f(x_0) \leq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute minimum of f on I
- **(iii)** If there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f
- **(iv)** If there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f
- **(v)** If there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative minimum of f
- **(vi)** If $f(x_0)$ is either a relative minimum or relative maximum of f, then $f(x_0)$ is a relative extremum of f

Remark 1.1. A relative extremum can only occur at an interior point, but an absolute extremum may occur at one of the end points of the interval. So if a function has an absolute maximum at a point x_0, it may not have a relative maximum at x_0. If f has an absolute maximum at an interior point x_0 of I, then $f(x_0)$ is also a relative maximum of f.

Lemma 1.1. Let $f : (a,b) \to \mathbb{R}$ and $f'(c)$ exists for some $c \in (a,b)$.

(i) If $f'(c) > 0$, then there exists $\delta > 0$ such that

$$f(x) < f(c) \text{ for every } x \in (c-\delta, c) \text{ and } f(x) > f(c) \text{ for every } x \in (c, c+\delta).$$

(ii) If $f'(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) > f(c) \text{ for every } x \in (c-\delta, c) \text{ and } f(x) < f(c) \text{ for every } x \in (c, c+\delta).$$

Theorem 1.3 (Fermat's extremum theorem)**.** Suppose c is an interior point of an interval I and $f : I \to \mathbb{R}$ is differentiable at c. If f has a relative extremum at c, then $f'(c) = 0$.

Proof. Without a loss of generality, assume that f has a relative maximum at c (the proof if f has a relative minimum is similar). Suppose on the contrary that either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then by the lemma above, there exists $\delta > 0$ such that $f(x) < f(c)$ for every $x \in (c-\delta, c)$ and $f(x) > f(c)$ for every $x \in (c, c+\delta)$. This contradicts the assumption that f has a relative maximum at c. The proofs for other cases are similar. □

Remark 1.2. A function f may have a relative extremum at x_0, but $f'(x_0)$ does not exist.

Example 1.2. Consider $f(x) = |x|$. There is a relative (absolute) minimum at $x = 0$, but $f'(0)$ does not exist.

The converse of Fermat's theorem is false. For example, consider $f(x) = x^3$, where $f'(0) = 0$ but $x = 0$ is not a relative extremum point of f. It is merely a point of inflection.

Theorem 1.4 (Rolle's theorem)**.** If f is continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Proof. The proof where $f(x)$ is a constant will not be discussed since it is trivial. For the more meaningful cases, we have $f(x) > f(a)$ or $f(x) < f(a)$ for some $x \in (a,b)$. Without a loss of generality, we shall prove the former case since the proof for the latter is similar.

By the extreme value theorem, we know that $f(x)$ has a maximum, M in the closed interval $[a,b]$. As $f(a) = f(b)$, the maximum value is attained at $x = c$. That is, $f(c) = M$. So, f has a local maximum at c. Since f is differentiable, the result follows. □

Theorem 1.5 (mean value theorem)**.** If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof. We wish to construct a function $g : [a,b] \to \mathbb{R}$ such that $g(a) = g(b) = 0$, with a point $c \in (a,b)$ such that $g'(c) = 0$. Suppose

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a} \cdot (x-a).$$

It is clear that g is continuous on $[a,b]$ and differentiable on (a,b), and $g(a) = g(b) = 0$. By Rolle's theroem, there exists $c \in (a,b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} = 0.$$

Rearranging the equation, we are done. □

Corollary 1.1. If f is continuous on $[a,b]$, differentiable on (a,b) and $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Example 1.3 (Berkeley Problems in Mathematics P1.1.25)**.** Let the function f from $[0,1]$ to $[0,1]$ have the following properties:

(i) f is C^1 (i.e. f is differentiable and its derivative is continuous)
(ii) $f(0) = f(1) = 0$
(iii) f' is non-increasing (i.e. f is concave down)

Prove that the arc length of the graph of f does not exceed 3.

Solution. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, by Rolle's theorem, there exists $c \in (0,1)$ such that $f'(c) = 0$. Since f is concave down, then f is increasing on $(0,c)$ and decreasing on $(c,1)$. On $[0,c]$, the arc length of f is given by

$$\int_0^c \sqrt{1+[f'(x)]^2}\,dx.$$

We shall partition $(0,c)$ into n equally sized subintervals. Hence, each interval has width c/n. So,

$$\int_0^c \sqrt{1+[f'(x)]^2}\,dx = \lim_{n\to\infty} \frac{c}{n}\sum_{k=1}^{n} \sqrt{1+[f'(\zeta_k)]^2} \quad \text{where } \zeta_k = \left(\frac{c(k-1)}{n}, \frac{ck}{n}\right).$$

By the mean value theorem, each ζ_k satisfies

$$f'(\zeta_k) = \frac{f(ck/n) - f(c(k-1)/n)}{c/n}$$

so

$$\begin{aligned}
\int_0^c \sqrt{1+[f'(x)]^2}\,dx &\le \lim_{n\to\infty} \frac{c}{n}\sum_{k=1}^{n} \sqrt{1+\left[\frac{f(ck/n) - f(c(k-1)/n)}{c/n}\right]^2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{c}{n}\right)^2 + \left[f\left(\frac{ck}{n}\right) - f\left(\frac{c(k-1)}{n}\right)\right]^2} \\
&\le \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{c}{n} + f\left(\frac{ck}{n}\right) - f\left(\frac{c(k-1)}{n}\right)\right] \quad \text{since } \sqrt{a^2+b^2} \le a+b \\
&= c + f(c) \quad \text{by method of difference.}
\end{aligned}$$

In a similar fashion, one can prove that

$$\int_c^1 \sqrt{1+[f'(x)]^2}\,dx \le 1 - c + f(c).$$

so

$$\int_0^1 \sqrt{1+[f'(x)]^2}\,dx \le c + f(c) + 1 - c + f(c) = 1 + 2f(c) \le 1 + 2 = 3$$

where we used the fact that the range of f is $[0,1]$. □

Proposition 1.4 (increasing and decreasing functions)**.** Let f be differentiable on (a,b).

(i) If $f'(x) \ge 0$ for all $x \in (a,b)$, then f is increasing on (a,b)

(ii) If $f'(x) \le 0$ for all $x \in (a,b)$, then f is decreasing on (a,b)

Theorem 1.6 (first derivative test). Let f be a continuous function on $[a,b]$ and $c \in (a,b)$. Suppose f is differentiable on (a,b) except possibly at c.

(i) If there is a neighbourhood $(c-\delta, c+\delta) \subseteq I$ of c such that $f'(x) \ge 0$ for $x \in (c-\delta, c)$ and $f'(x) \le 0$ for $x \in (c, c+\delta)$, then $f(c) \ge f(x)$ $\forall x \in (c-\delta, c+\delta)$. Hence, f has a relative maximum at c

(ii) If there is a neighbourhood $(c-\delta, c+\delta) \subseteq I$ of c such that $f'(x) \le 0$ for $x \in (c-\delta, c)$ and $f'(x) \ge 0$ for $x \in (c, c+\delta)$, then $f(c) \le f(x)$ $\forall x \in (c-\delta, c+\delta)$. Hence, f has a relative minimum at c

Consider a function $f(x)$. Its first derivative is denoted by $f'(x)$, second derivative is denoted by $f''(x) = f^{(2)}(x)$, and so on. In general, for $n \in \mathbb{N}$, the n^{th} derivative of f at c is defined as

$$f^{(n)}(c) = (f^{n-1})'(c).$$

Let I be an interval. Then, for $n \in \mathbb{N}$, $\mathcal{C}^n(I)$ is defined to be the set of functions f such that $f^{(n)}$ exists and is continuous on I. Note that

$$\mathcal{C}^\infty(I) = \bigcap_{n=1}^{\infty} \mathcal{C}^n(I).$$

If $\in \mathcal{C}^\infty(I)$, then f is infinitely differentiable on I.

Proposition 1.5. For $m > n \ge 1$, where $m, n \in \mathbb{Z}$,

$$\mathcal{C}^\infty(I) \subseteq \mathcal{C}^m(I) \subseteq \mathcal{C}^n(I) \subseteq \mathcal{C}(I).$$

Theorem 1.7 (second derivative test). Let f be defined on an interval I and let its derivative f' exist on I. Suppose c is an interior point of f such that $f'(c) = 0$ and $f''(c)$ exists.

(i) If $f''(c) > 0$, then f has a relative minimum at c

(ii) If $f''(c) < 0$, then f has a relative maximum at c

(iii) If $f''(c) = 0$, then the test is inconclusive. Hence, we have to use the first derivative test to prove whether c is a relative minimum, relative maximum, or a point of inflection

Theorem 1.8 (Cauchy's mean value theorem). Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \ne 0$ for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. We first claim that $g(a) \ne g(b)$. Suppose otherwise, then $g(a) = g(b)$, so by Rolle's theorem, there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$, contradicting the assumption that $g'(x) \ne 0$ for all $x \in (a,b)$. Next, define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)} \cdot ((g(x)-g(a)) - (f(x)-f(a)),$$

where $x \in [a,b]$. Since h is continuous on $[a,b]$, differentiable on (a,b) and $h(a) = h(b) = 0$, by Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. The result follows. $\square$

Theorem 1.9 (Taylor's theorem). Let f be a function such that $f \in \mathcal{C}^n([a,b])$ and $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$, then for any $x \in [a,b]$, there exists a point c between x and x_0 such that

$$f(x) = \sum_{k=0}^{n+1} \frac{f^k(c)}{k!}(x-x_0)^k.$$

Corollary 1.2. If $n = 0$, then $f(x) = f(x_0) + f'(c)(x-x_0)$, which is the mean value theorem.

The polynomial $P_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

is the n^{th} Taylor polynomial for f at x_0.

By Taylor's theorem, as $f(x) = P_n(x) + R_n(x)$, then

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1}$$

for some point c_n between x and x_0. This formula for R_n is the Lagrange form of the remainder.

Let f be infinitely differentiable on $I = (x_0 - r, x_0 + r)$ and $x \in I$. Then, recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

if and only if

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1} = 0,$$

where each c_n is between x and x_0.

Chapter 2
The Riemann–Stieltjes Integral

2.1
Definition and Existence

Let $I=[a,b]$. A finite set $P=\{x_0,x_1,x_2,\ldots,x_n\}$ where

$$a<x_0<x_1<x_2<\ldots<x_n<b$$

is a partition of I. It divides I into the subintervals as

$$I=[x_0,x_1]\cup[x_1,x_2]\cup[x_2,x_3]\cup\ldots\cup[x_{n-1},x_n]=\bigcup_{i=1}^{n}[x_{i-1},x_i].$$

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and let $P=\{x_0,x_1,x_2,\ldots,x_n\}$ be a partition of $[a,b]$. For each $1\le i\le n$, let

$$M_i=M_i(f,P)=\sup\{f(x):x\in[x_{i-1},x_i]\},$$

$$m_i=m_i(f,P)=\inf\{f(x):x\in[x_{i-1},x_i]\}\text{ and}$$

$\Delta x_i=x_i-x_{i-1}$. Define the upper sum and lower sum of f with respect to P to be

$$U(f,p)=\sum_{i=1}^{n}M_i\Delta x_i\text{ and }L(f,p)=\sum_{i=1}^{n}m_i\Delta x_i.$$

Note that each partition may not be of uniform length.

By setting $m=\inf\{f(x):x\in[a,b]\}$ and $M=\sup\{f(x):x\in[a,b]\}$, then

$$m(b-a)\le L(f,p)\le U(f,p)\le M(b-a).$$

Furthermore,

$$m(b-a)\le\int_a^b f\le M(b-a)$$

and if $f(x)\ge 0$ for all $x\in[a,b]$, then

$$\int_a^b f\ge 0.$$

Definition 2.1 (Darboux integral). The upper Darboux integral of f on $[a,b]$ is defined to be

$$U(f)=\overline{\int_a^b}f(x)\,dx=\inf\{U(f,P):P\text{ is a partition of }[a,b]\}$$

and the lower Darboux integral of f on $[a,b]$ is defined to be

$$L(f)=\underline{\int_a^b}f(x)\,dx=\sup\{L(f,P):P\text{ is a partition of }[a,b]\}.$$

Lemma 2.1. $L(f)\le U(f)$

Proof. We prove by contradiction. Suppose $U(f) < L(f)$. Then, there exists a partition P_1 of $[a,b]$ such that

$$U(f) \leq U(f,P_1) < L(f).$$

Also, there exists a partition P_2 of $[a,b]$ such that

$$U(f) \leq U(f,P_1) < L(f,P_2) \leq L(f).$$

However, $L(f,P_2) \leq U(f,P_1)$, which is a contradiction. □

From Lemma 2.1, it is clear that for partitions P and Q of $[a,b]$,

$$L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q),$$

and consequently,

$$L(f) = \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(f).$$

Definition 2.2. If P and Q are partitions of $[a,b]$, then Q is a refinement of P if $P \subseteq Q$.

Proposition 2.1. If P and Q are partitions of $[a,b]$ with Q a refinement of P, then

$$L(f,P) \leq L(f,Q) \quad \text{and} \quad U(f,Q) \leq U(f,P).$$

Definition 2.3. A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

The Riemann integral is only defined for bounded functions (i.e. if f is unbounded on $[a,b]$, it is not integrable on $[a,b]$) .

Example 2.1.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

is not integrable since $\lim_{x\to 0} 1/x^2 = \infty$, implying that the function is unbounded on $[-1,1]$.

Consider the Dirichlet function and denote it by $f(x)$. Since the rational and irrational numbers both form dense subsets of $\mathbb{R}$, then f takes on the value of 0 and 1 on every sub-interval of any partition. Thus for any partition P, $U(f,P) = 1$ and $L(f,P) = 0$. By noting that the upper and lower Darboux integrals are unequal, we conclude that f is not Riemann integrable on $[0,1]$[†].

2.2 Riemann Integrability Criterion and Consequences

Theorem 2.1 (Riemann integrability criterion)**.** For a bounded function $f : [a,b] \to \mathbb{R}$, then f is integrable on $[a,b]$ if and only if for any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

[†]A fun fact is that the Dirichlet function is actually Lebesgue integrable (covered in MA4262).

Proof. We first prove that if $U(f,P)-L(f,P)<\varepsilon$, then f is integrable on $[a,b]$. Note that $\varepsilon>0$ is arbitrary. Recall that

$$L(f,P)\le L(f)\le U(f)\le U(f,P).$$

Hence,

$$U(f)-L(f)\le U(f,P)-L(f,P)<\varepsilon,$$

and we are done.

Now, suppose f is integrable on $[a,b]$. We wish to prove that $U(f,P)-L(f,P)<\varepsilon$. Note that there exists a partition P_1 on $[a,b]$ such that $U(f,P_1)<U(f)$ so

$$U(f,P_1)<U(f)+\frac{\varepsilon}{2}.$$

In a similar fashion, there exists a partition P_2 such that

$$L(f,P_2)>L(f)-\frac{\varepsilon}{2}.$$

Let $P=P_1\cup P_2$ be the common refinement of the previous two partitions. Since

$$0\le U(f,P)-L(f,P),$$

then

$$0\le U(f,P)-L(f,P)<U(f)-L(f)+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon.$$

□

Corollary 2.1. If $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$, then f is integrable on $[a,b]$.

Corollary 2.2. If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Corollary 2.3. Let $f,g:[a,b]\to\mathbb{R}$ be bounded functions, P be a partition of $[a,b]$ and $c\in\mathbb{R}$. Then,

(i)

$$L(cf,P)=\begin{cases} cL(f,P) & \text{if } c>0 \\ cU(f,P) & \text{if } c<0 \end{cases}$$

(ii)

$$U(cf,P)=\begin{cases} cU(f,P) & \text{if } c>0 \\ cL(f,P) & \text{if } c<0 \end{cases}$$

(iii)

$$L(f,P)+L(g,P)\le L(f+g,P)\le U(f+g,P)\le U(f,P)+U(g,P)$$

Proposition 2.2. Let $f,g:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $c\in\mathbb{R}$. Then,

(i) Just like linear transformations, the function $cf+g$ is integrable on $[a,b]$ and

$$\int_a^b (cf+g)=c\int_a^b f+\int_a^b g.$$

(ii) If $f(x)\le g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f\le\int_a^b g.$$

(iii) $|f|$ is integrable on $[a,b]$ and
$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

(iv) fg is integrable on $[a,b]$.

Proposition 2.3. If f is integrable on $[a,b]$, then for any $c \in (a,b)$, f is integrable on $[a,c]$ and $[c,b]$. The converse is true and we have the following result:
$$\int_a^b f = \int_a^c f + \int_c^b f$$

Let f be a continuous function on $[a,b]$. If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a,b]$, define
$$L(f,P) = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$
where the supremum is taken over all possible partitions $a = x_0 < x_1 < x_2, \ldots < x_n = b$. This definition as the supremum of the all possible partition sums is also valid if f is merely continuous, not differentiable.

2.3
Fundamental Theorems of Calculus

Theorem 2.2 (First Fundamental Theorem of Calculus)**.** Let f be integrable on $[a,b]$ and for $x \in [a,b]$, let
$$F(x) = \int_a^x f.$$
If f is continuous at a point $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Remark 2.1. Not all functions have an elementary antiderivative. That is, for example, there do not exist elementary functions $F(x)$ and $G(x)$ such that
$$F(x) = \int e^{-x^2}\,dx \text{ and } G(x) = \int \frac{1}{\ln x}.$$

Theorem 2.3 (Second Fundamental Theorem of Calculus)**.** Let g be a differentiable function on $[a,b]$ and assume that g' is continuous on $[a,b]$. Then,
$$\int_a^b g' = g(b) - g(a).$$

Theorem 2.4 (Cauchy's Fundamental Theorem of Calculus)**.** Let g be a differentiable function on $[a,b]$ and assume that g' is integrable on $[a,b]$. Then,
$$\int_a^b g' = g(b) - g(a).$$

Example 2.2. It is possible for the derivative of a function to not be integrable. Consider the following function:

$$f(x) = \begin{cases} x^2 \sin\left(1/x^2\right) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

For $x \neq 0$,

$$f'(x) = -\frac{2}{x}\cos\left(\frac{1}{x^2}\right) + 2x\sin\left(\frac{1}{x^2}\right)$$

but $f'(x)$ is not integrable on $[-1,1]$ as this is a region of oscillating discontinuity!

2.4
Riemann Sum

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$ and let $\Delta x = x_i - x_{i-1}$ for $1 \leq i \leq n$. Then, the norm of P, denoted by $\|P\|$, is defined by

$$\|P\| = \max\left\{\Delta x_i : 1 \leq i \leq n\right\}.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$, $\|P\| < \delta$ implies that

$$U(f,P) < \overline{\int_a^b} f + \varepsilon \text{ and } L(f,P) > \underline{\int_a^b} f - \varepsilon.$$

We are now ready to define the Riemann sum of f with respect to P.

Definition 2.4 (Riemann sum). Let ξ_i be a point in the i^{th} sub-interval $[x_{i-1}, x_i]$ for $1 \leq i \leq n$. The sum

$$S(f,P)(\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is the Riemann sum of f with respect to P and $\xi = (\xi_1, \ldots, \xi_n)$.

If there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$ and any choice of $\xi = (\xi_1, \ldots, \xi_n)$,

$$\|P\| < \delta \quad \text{implies} \quad |S(f,P)(\xi) - A| < \varepsilon,$$

then

$$\lim_{\|P\| \to 0} S(f,P)(\xi) = A.$$

Note that

$$L(f,P) \leq S(f,P)(\xi) \leq U(f,P).$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then,

$$\lim_{\|P\| \to 0} U(f,P) = \overline{\int_a^b} f \quad \text{and} \quad \lim_{\|P\| \to 0} L(f,P) = \underline{\int_a^b} f.$$

Hence, f is integrable on $[a,b]$ and $\int_a^b f = A$ if and only if

$$\lim_{\|P\| \to 0} S(f,P)(\xi) = A.$$

Corollary 2.4. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. For each $n \in \mathbb{N}$, let $P_n = \left\{x_0^{(n)}, x_1^{(n)}, \dots, x_{m_n}^{(n)}\right\}$ be a partition of $[a,b]$ and let $\xi^{(n)} = \left(\xi_1^{(n)}, \dots, \xi_{m_n}^{(n)}\right)$ be such that $\xi_i^{(n)} \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right]$ for all $1 \le i \le m_n$. Define the sequence y_n as follows:

$$y_n = S(f,p)(\xi^{(n)})$$

If $\lim_{n\to\infty} \|P_n\| = 0$, then

$$\lim_{n\to\infty} y_n = \int_a^b f.$$

2.5 Improper Integrals

Definition 2.5 (improper integral)**.** An improper integral is one such that either the integrand, f, is unbounded on (a,b) or the interval of integration is unbounded.

Proposition 2.4. Suppose f is defined on $[a,b)$ and f is integrable on $[a,c]$ for every $c \in (a,b)$. If the limit

$$L = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

exists, then

$$\text{the improper integral } \int_a^b f(x)\,dx \text{ converges} \quad \text{and} \quad \int_a^b f(x)\,dx = L.$$

If the limit does not exist, then the improper integral diverges.

Similarly, if f is defined on $(a,b]$ and f is integrable on $[c,b]$ for every $c \in (a,b)$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} f(x)\,dx \quad \text{provided that the limit exists.}$$

Chapter 3
Sequences and Series of Functions

3.1
Pointwise and Uniform Convergence

An example of a sequence of functions $f_n(x)$, where $n \in \mathbb{N}$ is

$$f_n(x) = \frac{x+x^n}{2+x^n}$$

for $x \in [0,1]$. Then, consider the integral

$$\int_0^{\frac{1}{2}} f_n(x).$$

As $n \to \infty$, what can be deduced?

This section deals with questions like these. To start off, we need to introduce the ideas of pointwise convergence and uniform convergence.

Definition 3.1 (pointwise convergence). Let E be a non-empty subset of $\mathbb{R}$. Suppose for each $n \in \mathbb{N}$, we have a function $f_n : E \to \mathbb{R}$. Then, f_n is a sequence of functions on E. For each $x \in E$, the sequence $f_n(x)$ of real numbers converges. Define the function $f : E \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in E.$$

Then, f_n converges to f pointwise on E, and so $f_n \to f$ pointwise on E.

Definition 3.2 (pointwise convergence). $f_n \to f$ pointwise on E if and only if for every $x \in E$ and for every $\varepsilon > 0$, there exists $K = K(\varepsilon, x) \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon.$$

Remark 3.1. If $f_n \to f$ pointwise on I and each f_n is continuous on I, then f is not necessarily continuous on I.

Example 3.1. Consider $f_n(x) = x^n$ for $x \in [0,1]$. Note that each f_n is continuous on $[0,1]$. However, f is not continuous at $x = 1$ since for $x \in [0,1)$, then

$$\lim_{n\to\infty} f_n(x) = 0$$

but for $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = 1.$$

Remark 3.2. If $f_n \to f$ pointwise on $[a,b]$ and each f_n is integrable on $[a,b]$, then

(1). f is not necessarily integrable on $[a,b]$

(2). the pointwise convergence

$$\int_a^b g_n \to \int_a^b g \quad \text{is not necessarily true.}$$

Remark 3.3. If $f_n \to f$ pointwise on $[a,b]$ and each f_n and f are differentiable on $[a,b]$, then $f_n' \to f'$ not necessarily pointwise on $[a,b]$.

Example 3.2. Consider $f_n(x) = \sin(nx)/\sqrt{n}, x \in \mathbb{R}$. $f(x) = 0$ for all $x \in \mathbb{R}$, and thus $f_n \to f$ pointwise on $\mathbb{R}$. As $f'(x) = \sqrt{n}\cos(nx)$, for each $n \in \mathbb{N}$, $f' = 0$, but $f_n' \to f'$ pointwise on $\mathbb{R}$. Then, $f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$, but $f'(0) = 0$.

Definition 3.3 (uniform convergence). A sequence of functions f_n converges uniformly to f on E if for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. In this case, $f_n \to f$ uniformly on E. We say that the sequence f_n of functions converges uniformly on E if there exists a function f such that f_n converges to f uniformly on E.

Definition 3.4 (uniform norm). Let $E \subseteq \mathbb{R}$ and let $\phi : E \to \mathbb{R}$ be a bounded function. The uniform norm of ϕ on E is defined as

$$\|\phi\|_E = \sup\{|\phi(x)| : x \in E\}.$$

Then, $|\phi(x)| \leq \|\phi\|_E$ for all $x \in E$.

Lemma 3.1. A sequence of functions f_n converges to f uniformly on E if and only if $\|f_n - f\|_E \to 0$.

Proposition 3.1 (Cauchy criterion). A sequence of functions f_n converges uniformly on E if and only if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_E < \varepsilon \quad \text{for all } m, n \geq K.$$

Proposition 3.2. The following hold:

(i) If f_n converges uniformly on E, then f_n converges pointwise on E

(ii) If f_n converges uniformly on E and $F \subseteq E$, then f_n converges uniformly on F

Proposition 3.3. If f_n converges uniformly to f on an interval I and each f_n is continuous at $x_0 \in I$, then f is continuous at x_0.

Corollary 3.1. If f_n converges uniformly to f on I and each f_n is continuous on I, then f is continuous on I. Hence,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x),$$

implying that we can interchange the order of the two limit operations.

Proposition 3.4. Suppose $f_n \to f$ uniformly on $[a,b]$ and each f_n is integrable on $[a,b]$. Then,
(i): f is integrable on $[a,b]$ and

(ii): for each $x_0 \in [a,b]$, the sequence of functions

$$F_n(x) = \int_{x_0}^{x} f_n(t)\, dt$$

converges uniformly to the function

$$F(x) = \int_{x_0}^{x} f(t)\, dt$$

on $[a,b]$. Hence,

$$\lim_{n\to\infty}\int_{x_0}^{x} f_n(t)\, dt = \int_{x_0}^{x}\lim_{n\to\infty} f_n(t)\, dt$$

and in particular,

$$\lim_{n\to\infty}\int_{a}^{b} f_n(t)\, dt = \int_{a}^{b} f(t)\, dt.$$

Proposition 3.5. Suppose f_n is a sequence of differentiable functions on $[a,b]$ such that

$$f_n(x_0) \text{ converges for some } x_0 \in [a,b] \quad \text{and} f_n' \text{ converges uniformly on } [a,b].$$

Then, f_n converges uniformly on $[a,b]$ to a differentiable function f and for $a \le x \le b$,

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

3.2 Infinite Series of Functions

If f_n is a sequence of functions on E, then

$$S = \sum_{n=1}^{\infty} f_n \quad \text{is} \quad \text{an infinite series of functions.}$$

For each $n \in \mathbb{N}$ and $x \in E$, the n^{th} partial sum of S is the function

$$S_n(x) = \sum_{i=1}^{n} f_i(x).$$

Proposition 3.6. The following hold:

(i) S converges pointwise to a function S on E if the sequence S_n of functions converges pointwise to S on E

(ii) S converges uniformly to a function S on E if the sequence S_n of functions converges uniformly to S on E

(iii) S converges absolutely on E if the series

$$\sum_{n=1}^{\infty} |f_n| \quad \text{converges pointwise on } E$$

Proposition 3.7 (Cauchy criterion)**.** Let f_n be a sequence of functions on E. Then,

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } E$$

if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\text{for all } n > m \geq K \quad \text{we have} \quad \left\| \sum_{i=m+1}^{n} f_i \right\|_E < E.$$

Corollary 3.2. If

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } E \quad \text{then} \quad f_n \to 0 \text{ uniformly on } E.$$

Theorem 3.1 (Weierstrass M-test)**.** Let f_n be a sequence of functions on E and M_n be a sequence of positive real numbers such that $\|f_n\|_E \leq M_n$ for all $n \in \mathbb{N}$. If

$$\sum_{n=1}^{\infty} M_n \text{ converges} \quad \text{then} \quad \sum_{n=1}^{\infty} f_n \text{ converges uniformly and absolutely on } E.$$

Example 3.3. We can prove that the series expansion of the exponential function can be uniformly convergent on any bounded subset $S \subseteq \mathbb{C}$.

Solution. Let $z \in \mathbb{C}$. Note that the series expansion of the complex exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Any bounded subset is a subset of some disc D_R of radius R centred on the origin on the complex plane. The Weierstrass M-test requires us to find an upper bound, M_n, on the terms of the series, with M_n independent of the position in the disc. Observe that

$$\left| \frac{z^n}{n!} \right| \leq \frac{|z|^n}{n!} \leq \frac{R^n}{n!}$$

so by setting $M = R^n/n!$, we are done. □

Proposition 3.8. If

$$\sum_{n=1}^{\infty} f_n \to f \quad \text{uniformly on } I$$

and each f_n is continuous on each $x_0 \in I$, then f is continuous at x_0.

We now state some properties related to differentiability and integrability.

Proposition 3.9. If

$$\sum_{n=1}^{\infty} f_n \to f \text{ uniformly on } [a,b] \quad \text{and} \quad \text{each } f_n \text{ is integrable on } [a,b],$$

(1). f is integrable on $[a,b]$

(2). for every $x \in [a,b]$,

$$\sum_{n=1}^{\infty} \int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt = \int_a^x \sum_{n=1}^{\infty} f_n(t)\, dt$$

where the convergence is uniform on $[a,b]$

Proposition 3.10. Suppose f_n is a sequence of differentiable functions on $[a,b]$ such that

$$\sum_{n=1}^{\infty} f_n(x_0) \text{ converges for some } x_0 \in [a,b] \quad \text{and} \quad \sum_{n=1}^{\infty} f_n' \text{ converges uniformly on } [a,b].$$

Then,

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } [a,b] \text{ to a differentiable function } f \quad \text{and}$$

$$\sum_{n=1}^{\infty} f_n'(x) = f'(x) \text{ for } a \le x \le b$$

Chapter 4
Power Series

4.1
Introduction

Definition 4.1 (power series). A series of functions of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

where $x_0, a_1, a_2, \ldots$ are constants, is a power series in $x - x_0$. So,

$$\sum_{n=0}^{\infty} (x-x_0)^n = \sum_{n=0}^{\infty} f_n(x)$$

where for each n, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = a_n(x-x_0)^n$.

If $x_0 = 0$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Proposition 4.1. Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{be} \quad \text{a power series.}$$

(i) If it converges at $x = x_1$, then it is absolutely convergent for all values of x for which $|x-x_0| < |x_1 - x_0|$

(ii) If it diverges for $x = x_2$, then it diverges for all values of x such that $|x - x_0| > |x_2 - x_0|$

4.2
Radius of Convergence

Definition 4.2 (radius of convergence). Given a power series, let

$$S = \left\{ |x-x_0| : x \in \mathbb{R} \quad \text{and} \quad \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges} \right\}$$

The radius of convergence of the series, R, is defined as follows:

(i) $R = 0$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges only for } x = x_0$$

(ii) $R = \infty$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for all } x \in \mathbb{R}$$

(iii) $R = \sup S$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for some } x \text{ and diverges for others}$$

Example 4.1. The series

$$\sum_{n=0}^{\infty} n!x^n$$

converges only at $x=0$, implying that $R=0$.

Example 4.2. The exponential function e^x converges at every point of $\mathbb{R}$, and so $R=\infty$.

Example 4.3. Consider the geometric series $1+x+x^2+x^3+\ldots$, which converges for all $x \in \mathbb{R}$ and diverges for all oher x's. Hence, $R=1$.

Definition 4.3 (absolute convergence).

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges absolutely}$$

for all $x \in (x_0-R, x_0+R)$ and diverges for all x with $|x-x_0|>R$.

Theorem 4.1 (ratio test). Suppose a_n is non-zero for all n. Let

$$\rho = \lim_{n=0} \left| \frac{a_{n+1}}{a_n} \right|.$$

(i) If the limit ρ exists, then the radius of convergence, R, of the power series is

$$R = \begin{cases} 1/\rho & \text{if } \rho > 0; \\ \infty & \text{if } \rho = 0 \end{cases}$$

(ii) If $\rho=\infty$, then $R=0$

The ratio test is frequently easier to apply than the root test since it is usually easier to evaluate ratios than n^{th} roots. However, the root test is a *stronger* test for convergence. This means that whenever the ratio test shows convergence, the root test does too and whenever the root test is inconclusive, the ratio test is too (merely the contrapositive statement).

For any sequence x_n of positive numbers,

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \text{ and } \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Theorem 4.2 (Cauchy-Hadamard formula). Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ be a power series} \quad \text{and} \quad \rho = \limsup |a_n|^{1/n}.$$

The radius of convergence, R, is

$$R = \begin{cases} 0 & \text{if } \rho = \infty; \\ 1/\rho & \text{if } 0 < \rho < \infty; \\ \infty & \text{if } \rho = 0 \end{cases}$$

4.3 Properties of Power Series

Proposition 4.2. Suppose

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

has a radius of convergence $R>0$. Then, f is infinitely differentiable on (x_0-R,x_0+R), i.e.

$$f'(x)=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1} \quad \text{where } x\in(x_0-R,x_0+R).$$

Proposition 4.3. For every $k\in\mathbb{N}$, we have the following result:

$$f^{(k)}(x)=\sum_{n=k}^{\infty}n(n-1)(n-2)\dots(n-k+1)a_n(x-x_0)^{n-k} \quad \text{where } x\in(x_0-R,x_0+R)$$

and the radius of convergence of each of these derived series is also R.

Although a power series and its derived series have the same values of R, they may converge on different sets.

Example 4.4. Consider the power series

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

By the ratio test, $R=1$, so the series converges in $(-1,1)$. The series also converges at $x=\pm1$. In fact, when $x=1$, we obtain the famous p-series for which $p=2$, and it is also known as the Basel problem. When $x=-1$, we obtain a variant of the Basel problem which can be evaluated as well. Hence, the series converges in $[-1,1]$.

Differentiating both sides of the power series gives

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n},$$

where $x\in(-1,1)$. $f'(x)$ converges at $x=-1$ but diverges at $x=1$, which is the harmonic series. Hence, $f'(x)$ converges on $[-1,1)$.

Corollary 4.1. If there exists $r>0$ such that

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n \quad \text{for all } x\in(x_0-r,x_0+r),$$

then

$$a_k=\frac{f^{(k)}(x_0)}{k!} \quad \text{for all } k\in\mathbb{Z}_{\geq0}.$$

Corollary 4.2 (uniqueness of power series)**.** If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

for all $x\in(x_0-r,x_0+r)$ for some $r>0$, then $a_n=b_n$ for all $n\in\mathbb{Z}_{\geq0}$.

Corollary 4.3. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{have} \quad \text{a non-zero radius of convergence } R.$$

Then, for any a and b for which $x_0 - R < a < b < x_0 + R$,

$$\int_a^b f(x)\,dx = \int_a^b \sum_{n=0}^{\infty} a_n(x-x_0)^n\,dx = \sum_{n=0}^{\infty} \int_a^b a_n(x-x_0)^n\,dx.$$

In other words, a power series can be integrated term-by-term over any closed interval $[a,b]$ contained in the interval of convergence.

Theorem 4.3 (Abel summation formula)**.** Let b_n and c_n be sequences of real numbers, and for each pair of integers $n \geq m \geq 1$, set

$$B_{n,m} = \sum_{k=m}^{n} b_k.$$

Then,

$$\sum_{k=m}^{n} b_k c_k = B_{n,m}c_n - \sum_{k=m}^{n-1} B_{k,m}(c_{k+1} - c_k).$$

for all $n > m \geq 1$, $n, m \in \mathbb{N}$.

Theorem 4.4 (Abel's theorem)**.** Suppose

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{has} \quad \text{a finite non-zero radius of convergence } R.$$

(i) If the series converges at $x = x_0 + R$, then it converges uniformly on $[x_0, x_0 + R]$

(ii) If the series converges at $x = x_0 - R$, then it converges uniformly on $[x_0 - R, x_0]$

4.4 Taylor Series

A function f is infinitely differentiable on (a,b) if $f^{(n)}(x)$ exists for all $x \in (a,b)$ and for all $n \in \mathbb{N}$. This class of functions is denoted by $\mathcal{C}^\infty$.

Definition 4.4 (Taylor series)**.** Let f be infinitely differentiable on $(x_0 - r, x_0 + r)$ for some $r > 0$. The power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \text{is} \quad \text{the Taylor series of } f \text{ about } x_0.$$

Definition 4.5 (Taylor series)**.** Considering the Taylor series, set $x_0 = 0$. We then obtain the Maclaurin Series of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 4.6 (analytic function). A function f is analytic on (a,b) if f is infinitely differentiable on (a,b) and for any $x_0 \in (a,b)$, the Taylor Series of f about x_0 converges to f in a neighbourhood of x_0.

Example 4.5. The functions e^x, $\sin x$ and $\cos x$ are analytic on $\mathbb{R}$ and the infinite geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is analytic on $(-1,1)$.

4.5 Arithmetic Operations with Power Series

Definition 4.7 (Cauchy product). The Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where for each $n \in \mathbb{N}$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0.$$

Proposition 4.4. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n,\ |x-x_0| < R_1 \text{ and } g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n,\ |x-x_0| < R_2.$$

For $\alpha, \beta \in \mathbb{R}$, we have the following:

(i)

$$\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n \text{ for } |x-x_0| < \min(R_1, R_2)$$

(ii)

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n,\ |x-x_0| < \min(R_1,R_2) \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Theorem 4.5 (Merten's theorem). If

$$\sum_{n=0}^{\infty} a_n \text{ converges absolutely and } \sum_{n=0}^{\infty} b_n \text{ converges} \quad \text{then} \quad \text{the Cauchy product } \sum_{n=0}^{\infty} c_n \text{ converges.}$$

Also,

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Recall that for Merten's theorem, we just need at least one of the series to converge absolutely.

Definition 4.8 (conditional convergence). A series is conditionally convergent if it converges but does not converge absolutely.

Remark 4.1. If

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n \quad \text{converge conditionally,}$$

then their Cauchy product may not converge.

Example 4.6. Set

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}},$$

where $n \geq 0$. It is clear that both series are conditionally convergent (but not absolutely convergent) by the alternating series test. The Cauchy product of these two series is

$$c_n = \sum_{k=0}^{\infty} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}} \quad \text{for all } n \in \mathbb{N}.$$

Note that $n \geq k$ so $n+1 \geq k+1$ and $n+1 \geq n-k+1$ so we are able to obtain a lower bound for $|c_n|$. Hence,

$$|c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1 \quad \text{which implies} \quad \sum_{n=0}^{\infty} c_n \text{ diverges.}$$

Theorem 4.6 (Riemann rearrangement theorem). Suppose a_n is a sequence of real numbers, and that

$$\sum_{n=1}^{\infty} a_n \quad \text{is} \quad \text{conditionally convergent.}$$

Let $M \in \mathbb{R}$. Then, there exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

There also exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \infty.$$

The sum can also be rearranged to diverge to $-\infty$ or to fail to approach any limit, finite or infinite.

4.6 Some Special Functions

Definition 4.9 (exponential function). The function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for all $x \in \mathbb{R}$, is the exponential function.

Proposition 4.5. $E : \mathbb{R} \to \mathbb{R}$ has the following properties:

(i) $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$

(ii) $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$

(iii) $E(x) > 0$ for all $x \in \mathbb{R}$

(iv) E is strictly increasing (i.e. $E'(x) > 0$ for all $x \in \mathbb{R}$)

(v)

$$\lim_{x\to\infty} E(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} E(x) = 0$$

For **(i)**, any function $f(x)$ that has this property is invariant under successive levels of differentiation. Actually, one can verify that the exponential function is indeed the only function that is invariant under the differential operator by treating the differential equation $f'(x) = f(x)$ as a separable one.

Proposition 4.6. The functional equation

$$f(x+y) = f(x)f(y)$$

holds true only for the exponential function.

Euler's number, e (≈ 2.71828459045) is defined as the following limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Proposition 4.7. By considering the Maclaurin series of e^x, setting $x = 1$ gives the expansion

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

Proposition 4.8. In relation to sequences, for $x \in \mathbb{R}$, e^x is defined as

$$e^x = \lim_{n\to\infty} e^{r_n},$$

where r_n is an increasing rational sequence which converges to x.

Proposition 4.9. For $x \in \mathbb{R}$, e^x is continuous on $\mathbb{R}$.

Since the exponential function E is strictly increasing on $\mathbb{R}$ and $E(\mathbb{R}) = (0,\infty)$, then it implies that E is injective and thus has an inverse function $L : (0,\infty) \to \mathbb{R}$, which is also strictly increasing.

We have the following composition of functions

$$L(E(x)) = x \ \forall x \in \mathbb{R}$$

and

$$E(L(y)) = y \ \forall y > 0.$$

Definition 4.10 (natural logarithm)**.** By the Fundamental Theorem of Calculus, we define $L(y)$ to be the following integral:

$$L(y) = \int_1^y \frac{1}{t}\,dt$$

The function $L : (0,\infty) \to \mathbb{R}$ is the natural logarithm, $\ln(x)$.

Proposition 4.10. The natural logarithm $\ln : (0,\infty) \to \mathbb{R}$ has the following properties:

(i)

$$\frac{d}{dy}\ln y = \frac{1}{y} \quad \text{for all } y > 0$$

(ii)

$$\ln y = \int_1^y \frac{1}{t}\, dt \quad \text{for all } y > 0$$

(iii) $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y, > 0$

(iv) $\ln(1) = 0$ and $\ln(e) = 1$

(v) For $x > 0$ and $\alpha \in \mathbb{R}$, $x^\alpha = e^{\alpha \ln x}$

Proposition 4.11. The functional equation

$$f(xy) = f(x) + f(y)$$

holds true only for the logarithmic function.

Corollary 4.4. Let $\alpha \in \mathbb{R}$. Then, the function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$f(x) = x^\alpha$$

for all $x > 0$ is differentiable on $(0, \infty)$ and

$$f'(x) = \alpha x^{\alpha - 1}$$

for all $x > 0$ as well.

Definition 4.11 (cosine)**.** The function

$$C(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$, is the cosine function.

Definition 4.12 (sine)**.** The function

$$S(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all $x \in \mathbb{R}$, is the sine function.

These two trigonometric functions have the following relationship, that for all $x \in \mathbb{R}$,

$$C'(x) = -S(x) \text{ and } S'(x) = C(x).$$

Differentiating both sides of each equation will yield

$$C''(x) = -C(x) \text{ and } S''(x) = -S(x),$$

which are second order linear homogeneous differential equations which constant coefficients.

Thus, we make the claim that if $g : \mathbb{R} \to \mathbb{R}$ has the property that $g''(x) = -g(x)$ for all $x \in \mathbb{R}$, then

$$g(x) = \alpha C(x) + \beta S(x)$$

for all $x \in \mathbb{R}$ too, where $\alpha = g(0)$ and $\beta = g'(0)$. The two functions satisfy the identity $(C(x))^2 + (S(x))^2 = 1$ for all $x \in \mathbb{R}$, which is also known as the Pythagorean identity.

The cosine function is even. That is, $C(-x) = C(x)$ (i.e. the graph is symmetrical about the y-axis). It satisfies the following addition formula:

$$C(x+y) = C(x)C(y) - S(x)S(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The sine function is odd. That is, $S(-x) = -S(x)$ (i.e. the graph is symmetrical about the origin). It satisfies the following addition formula:

$$S(x+y) = S(x)C(y) + C(x)S(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The four other trigonometric functions, as well as all the inverse trigonometric functions, will not be discussed. Moreover, respective small angle approximations will not be discussed too.

Definition 4.13 (gamma function). The gamma function is one commonly used extension of the factorial function to complex numbers. Denoted by $\Gamma(z)$, it is defined to be the following convergent improper integral:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt \quad \text{where } \Re(z) > 0.$$

Theorem 4.7. For $z \geq 0$, we have the following relationship:

$$\Gamma(z+1) = z\Gamma(z),$$

which has some semblance to the functional equation $f(x+1) = xf(x)$. Hence,

$$\Gamma(n) = (n-1)! \quad \text{where } n \in \mathbb{N}.$$

Proof. Using integration by parts,

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt = -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}\,dt = z\Gamma(z)$$

and we are done.

Next, to prove the closed form for $\Gamma(n)$, as $\Gamma(1) = 1$, so

$$\begin{aligned}\prod_{i=1}^{n-1}\frac{\Gamma(i+1)}{\Gamma(i)} &= \prod_{i=1}^{n-1} i \\ \frac{\Gamma(n)}{\Gamma(1)} &= (n-1)!\end{aligned}$$

and the result follows by the telescoping product. □

Theorem 4.8. $\ln\Gamma(z)$ is convex on $(0, \infty)$

Theorem 4.9 (Bohr-Mollerup theorem). The gamma function is the only function satisfying $f(1) = 1$, $f(x+1) = xf(x)$ and f is logarithmically convex.

The Bohr-Mullerup theorem characterises the gamma function.

There are two types of Euler integral. The gamma function is also known as the Euler integral of the first kind and the beta function (discussed in the next section) is also known as the Euler integral of the second kind.

Euler's reflection formula and Legendre's duplication formula are examples of functional equations closely related to the gamma function.

Theorem 4.10 (Euler's reflection formula). For $z \notin \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z).$$

Theorem 4.11 (Legendre duplication formula).

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Definition 4.14 (beta function). For $x, y \in \mathbb{C}$, where $\Re(x) > 0$ and $\Re(y) > 0$, the Beta Function $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

Proposition 4.12. $B(x,y)$ is symmetric.

Proof. Use the substitution $x = y$ so $B(x,y) = B(y,x)$, □

Theorem 4.12. The beta function is closely related to the gamma function and the binomial coefficients by the following equation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

The proof of Theorem 4.12 hinges on writing $\Gamma(x)\Gamma(y)$ as a double integral and using the technique of change of variables.

Chapter 5
Functions of Several Variables